Billiard representation for multidimensional multi-scalar cosmological model with exponential potentials

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Abstract

Multidimensional cosmological-type model with n Einstein factor spaces in the theory with l scalar fields and multiple exponential potential is considered. The dynamics of the model near the singularity is reduced to a billiard on the $(N-1)$-dimensional Lobachevsky space H^{N-1}, $N = n + l$. It is shown that for $n > 1$ the oscillating behaviour near the singularity is absent and solutions have an asymptotical Kasner-like behavior. For the case of one scale factor $(n = 1)$ billiards with finite volumes (e.g. coinciding with that of the Bianchi-IX model) are described and oscillating behaviour of scalar fields near the singularity is obtained.

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1 Introduction

The study of different aspects of multidimensional models in gravitation and cosmology in arbitrary dimensions and with sources as fluids and fields we started more than a decade ago (see [1, 2, 3]). Special attention was devoted to the treatment of dilatonic interactions with electromagnetic fields and fields of forms of arbitrary rank [4]. Here we continue our investigations of multidimensional models, in particular with multiple exponential potential (MEP) [5] (for $D = 4$ case see [6]).

The models of such sort are currently rather popular (see, for example, [6, 7, 8, 9, 10] and refs. therein). Such potentials arise naturally in certain supergravitational models [10], in sigma-models [11] related to configurations with p-branes and in reconstruction from observations schemes [12]. They also appear when certain $f(R)$ generalizations of Einstein-Hilbert action are considered [13].

Like in [5], here we consider D-dimensional model governed by the action

$$S_{act} = \int_M d^D Z \sqrt{|g|} \{ R[g] - h_{\alpha\beta} g^{MN} \partial_M \varphi^\alpha \partial_N \varphi^\beta - 2V_\varphi(\varphi) \} + S_{GH}, \quad (1.1)$$

$D > 2$, with the scalar potential (MEP)

$$V_\varphi(\varphi) = \sum_{s \in S} \Lambda_s \exp[2\lambda_s(\varphi)]. \quad (1.2)$$

$S_{\rm GH}$ is the standard Gibbons-Hawking boundary term [14].

The notations used are the following ones:

* $\varphi = (\varphi^\alpha)$ is the vector from scalar fields in the space $\mathbb{R}^l$ with a metric determined by a non-degenerate $l \times l$ matrix $(h_{\alpha\beta})$ with the inverse one $(h^{\alpha\beta})$; $\alpha, \beta = 1, \ldots, l$;
* Λ_s are constant terms; $s \in S$;
* λ_s is an 1-form on $\mathbb{R}^l$: $\lambda_s(\varphi) = \lambda_{s\alpha}\varphi^\alpha$; $\lambda_s^\alpha = h^{\alpha\beta}\lambda_{s\beta}$;
* $g = g_{MN} dZ^M \otimes dZ^N$ is a metric, $|g| = |\det(g_{MN})|$,
  M, N are world indices that may be numerated by $1, \ldots, D$.
* $i, j = 1, \ldots, n$ are indices describing a chain of factor spaces;
  $A = i, \alpha$ and $B = j, \beta$ are minisuperspace indices,
  that may be numerated also by $1, \ldots, n + l$.

This paper is devoted to the investigation of the possible oscillating (and probably stochastic) behaviour near the singularity (see [15]-[36] and references therein) for cosmological type solutions corresponding to the action (1.1).

We remind that near the singularity one can have an oscillating behavior like in the well-known mixmaster (Bianchi-IX) model [15]-[17] (see also [26]-[28]). Multidimensional generalizations and analogues of this model were considered by many authors (see, for example, [18]-[25]). In [29, 30, 31] a billiard representation for multidimensional cosmological models near the singularity was considered and the criterion for a volume of the billiard to be finite was established in terms of illumination of the unit sphere by point-like sources. For multicomponent perfect-fluid this was considered in detail in [31] and generalized to p-brane case in [34] (see also [36] and refs. therein). Some topics related to general (non-homogeneous) situation were considered in [32, 33].

Here we apply the billiard approach suggested in [29, 30, 31] to a cosmological model with MEP. We show that (as for the exact solutions from [5]) for $n > 1$ the oscillating behaviour near the singularity is absent. For $n = 1$ we find here examples of oscillating behavior for scalar fields but not for a scale factor.

The paper is organized as follows. In Sec. 2 the cosmological model with MEP is considered: Lagrange representation to equations of motion and the diagonalization of the Lagrangian are presented. In Sec. 3 a billiard approach in the multidimensional cosmology with MEP is obtained, and the case $n > 1$ is studied. Sec. 4 is devoted to description of billiards with finite volumes in the case of one scale factor ($n = 1$).

2 The model

Let

$$M = \mathbb{R} \times M_1 \times \ldots \times M_n \tag{2.1}$$

be a manifold equipped with the metric

$$g = w e^{2\gamma(u)} du \otimes du + \sum_{i=1}^{n} e^{2\phi^i(u)} g^i, \tag{2.2}$$

where $w = \pm 1$, u is a distinguished coordinate; g^i is a metric on d_i-dimensional manifold M_i, obeying:

$$\mathrm{Ric}[g^i] = \xi_i g^i, \tag{2.3}$$

$\xi_i = \mathrm{const}$, $i = 1, \ldots, n$. Thus, (M_i, g^i) are Einstein spaces.

For dilatonic scalar fields we put

$$\varphi^\alpha = \varphi^\alpha(u), \tag{2.4}$$

2.1 Lagrangian representation

It may be verified that the equations of motion (see Appendix A) corresponding to (1.1) for the field configuration (2.2)-(2.4) are equivalent to equations of motion for 1-dimensional σ-model with the action

$$S_\sigma = \frac{1}{2} \int du \mathcal{N} \Big\{ G_{ij} \dot{\phi}^i \dot{\phi}^j + h_{\alpha\beta} \dot{\varphi}^\alpha \dot{\varphi}^\beta - 2\mathcal{N}^{-2} V \Big\}, \tag{2.5}$$

where $\dot{x} \equiv dx/du$,

$$V = -w V_\varphi(\varphi) \mathrm{e}^{2\gamma_0(\phi)} + \frac{w}{2} \sum_{i=1}^{n} \xi_i d_i \mathrm{e}^{-2\phi^i + 2\gamma_0(\phi)} \tag{2.6}$$

is the potential (V_φ is defined in (1.2)) with

$$\gamma_0(\phi) \equiv \sum_{i=1}^{n} d_i \phi^i, \tag{2.7}$$

and

$$\mathcal{N} = \exp(\gamma_0 - \gamma) > 0 \tag{2.8}$$

is the lapse function. Here

$$G_{ij} = d_i \delta_{ij} - d_i d_j, \qquad G^{ij} = \frac{\delta^{ij}}{d_i} + \frac{1}{2 - D}, \tag{2.9}$$

$i, j = 1, \ldots, n$, are components of a gravitational part of minisupermetric and its dual [40].

2.2 Minisuperspace notations

In what follows we consider minisuperspace $\mathbb{R}^{n+l}$ with points

$$x \equiv (x^A) = (\phi^i, \varphi^\alpha) \tag{2.10}$$

equipped by minisuperspace metric $\bar{G} = \bar{G}_{AB} dx^A \otimes dx^B$ defined by the matrix and inverse one as follows:

$$(\bar{G}_{AB}) = \begin{pmatrix} G_{ij} & 0 \\ 0 & h_{\alpha\beta} \end{pmatrix}, \quad (\bar{G}^{AB}) = \begin{pmatrix} G^{ij} & 0 \\ 0 & h^{\alpha\beta} \end{pmatrix}. \tag{2.11}$$

The potential (2.6) reads

$$V = -w \sum_{s \in S} \Lambda_s \mathrm{e}^{2U^s(x)} + \sum_{j=1}^{n} \frac{w}{2} \xi_j d_j \mathrm{e}^{2U^j(x)}, \tag{2.12}$$

where $U^s(x) = U^s_A x^A$ and $U^j(x) = U^j_A x^A$ are defined as

$$U^s(x) = \lambda_{s\alpha} \varphi^\alpha + \gamma_0(\phi), \tag{2.13}$$

$$U^j(x) = -\phi^j + \gamma_0(\phi), \tag{2.14}$$

or, in components,

$$(U^s_A) = (d_i, \lambda_{s\alpha}) \tag{2.15}$$

$$(U^j_A) = (-\delta^j_i + d_i, 0) \tag{2.16}$$

$s \in S$; $i, j = 1, \dots, n$.

The integrability of the Lagrange system (2.5) depends upon the scalar products of co-vectors U^s, U^i corresponding to $\bar{G}$:

$$(U, U') = \bar{G}^{AB} U_A U'_B, \tag{2.17}$$

These products have the following form

$$(U^i, U^j) = \frac{\delta_{ij}}{d_j} - 1, \tag{2.18}$$

$$(U^s, U^{s'}) = -b + \lambda_s \cdot \lambda_{s'}, \tag{2.19}$$

$$(U^s, U^i) = -1, \tag{2.20}$$

where

$$\lambda_s \cdot \lambda_{s'} \equiv \lambda_{s\alpha} \lambda_{s'\beta} h^{\alpha\beta}, \quad b = \frac{D-1}{D-2}, \tag{2.21}$$

$s, s' \in S$.

2.3 Diagonalization of the Lagrangian

Let the matrix $(h_{\alpha\beta})$ have the Euclidean signature. Then, the minisuperspace metric (2.11) has a pseudo-Euclidean signature $(-,+,\ldots,+)$ since the matrix (G_{ij}) has the pseudo-Euclidean signature [40].

Hence there exists a linear transformation

$$z^a = e^a_A x^A, \tag{2.22}$$

diagonalizing the minisuperspace metric (2.11)

$$\bar{G} = \eta_{ac} dz^a \otimes dz^c = -dz^0 \otimes dz^0 + \sum_{k=1}^{N-1} dz^k \otimes dz^k, \tag{2.23}$$

where

$$(\eta_{ac}) = (\eta^{ac}) \equiv \mathrm{diag}(-1, +1, \ldots, +1), \tag{2.24}$$

and here and in what follows $a, c = 0, \ldots, N-1$; $N = n + l$. The matrix of linear transformation (e^a_A) satisfies the relation

$$\eta_{ac} e^a_A e^c_B = \bar{G}_{AB} \tag{2.25}$$

or, equivalently,

$$\eta^{ac} = e^a_A \bar{G}^{AB} e^c_B = (e^a, e^c), \tag{2.26}$$

where $e^a = (e^a_A)$.

Inverting the map (2.22) we get

$$x^A = e^A_a z^a, \tag{2.27}$$

where for components of the inverse matrix $(e^A_a) = (e^a_A)^{-1}$ we obtain from (2.26)

$$e^A_a = \bar{G}^{AB} e^c_B \eta_{ca}. \tag{2.28}$$

Like in [31] we put

$$e^0 = q^{-1} U^\Lambda, \qquad q = [-(U^\Lambda, U^\Lambda)]^{1/2} = b^{1/2}. \tag{2.29}$$

where $U^\Lambda(x) = U^\Lambda_A x^A = \gamma_0(\phi)$ is co-vector corresponding to the cosmological term, or, in components

$$(U^\Lambda_A) = (d_i, 0), \tag{2.30}$$

and hence

$$z^0 = e^0_A x^A = \sum_{i=1}^{n} q^{-1} d_i x^i. \tag{2.31}$$

In z-coordinates (2.22) with z^0 from (2.31) the Lagrangian corresponding to (2.5) reads

$$L = L(z, \dot{z}, \mathcal{N}) = \frac{1}{2} \mathcal{N}^{-1} \eta_{ac} \dot{z}^a \dot{z}^c - \mathcal{N} V(z), \tag{2.32}$$

where

$$V(z) = \sum_{r \in S_*} A_r \exp(2 u^r_a z^a) \tag{2.33}$$

is a potential,

$$S_* = \{1, \ldots, n\} \cup S \tag{2.34}$$

is an extended index set and

$$A_j = \frac{w}{2} \xi_j d_j, \quad A_s = -w \Lambda_s, \tag{2.35}$$

$j = 1, \ldots, n$; $s \in S$. Here we denote

$$u^r_a = e^A_a U^r_A = (U^r, e^c) \eta_{ca}, \tag{2.36}$$

$a = 0, \ldots, N-1$; $r \in S_*$ (see (2.28)).

From (2.17), (2.29) and (2.36) we deduce

$$u^r_0 = -(U^r, e^0) = (\sum_{i=1}^{n} U^r_i)/q(D-2), \tag{2.37}$$

$r \in S_*$.

For the potential-term and curvature-term components we obtain from (2.29) and (2.37)

$$u^s_0 = q > 0, \qquad u^j_0 = 1/q > 0, \tag{2.38}$$

$j = 1, \ldots, n$.

We remind that (see (2.18))

$$(U^j, U^j) = \left(\frac{1}{d_j} - 1 \right) < 0, \tag{2.39}$$

for $d_j > 1$, $j = 1, \ldots, n$. For $d_j = 1$ we have $\xi^j = A_j = 0$.

3 Billiard representation

Here we put the following restriction on parameters of the model:

$$- w\Lambda_s > 0, \tag{3.1}$$

$$\text{if} \quad (U^s, U^s) = -b + \lambda_s^2 > 0, \tag{3.2}$$

$s \in S$. In what follows we denote by S_+ a subset of all $s \in S$ satisfying (3.2). As we shall see below these restrictions are necessary for a formation of billiard "walls" (with positive infinite potential) in approaching to singularity.

Due to relations (2.35), (2.38), (2.39) and (3.1) the parameters u_a^r in the potential (2.33) obey the following restrictions:

$$\mathbf{1}.\ A_r > 0 \text{ for } (u^r)^2 = -(u_0^r)^2 + (\vec{u}^r)^2 > 0; \tag{3.3}$$

$$\mathbf{2}.\ u_0^r > 0 \text{ for all } r \in S_*. \tag{3.4}$$

Due to relations (3.3) and (3.4) the Lagrange system (2.32) for $N \geq 3$ in the ("near the singularity") limit

$$z^0 \to -\infty, \qquad z^0 < -|\vec{z}|, \tag{3.5}$$

may be reduced to a motion of a point-like particle in $N-1$-dimensional billiard belonging to Lobachevsky space [29, 30, 31, 34].

For non-exceptional asymptotics (non-Milne-type) the limit (3.5) describes the approaching to the singularity. (in this case the volume scale factor vanishes $\exp(\sum_{i=1}^n d_i x^i) = \exp(qz^0) \to +0$).

Indeed, introducing generalized Misner-Chitre coordinates in the lower light cone $z^0 < -|\vec{z}|$ [29, 30]

$$z^0 = -\exp(-y^0)\frac{1+\vec{y}^2}{1-\vec{y}^2}, \tag{3.6}$$

$$\vec{z} = -2\exp(-y^0)\frac{\vec{y}}{1-\vec{y}^2}, \tag{3.7}$$

$|\vec{y}| < 1$, and fixing the time gauge

$$\mathcal{N} = \exp(-2y^0) = -z^2. \tag{3.8}$$

we get in the limit $y^0 \to -\infty$ (after separating y^0 variable)

a "billiard" Lagrangian

$$L_B = \frac{1}{2}\bar{h}_{ij}(\vec{y})\dot{y}^i\dot{y}^j - V(\vec{y}, B). \tag{3.9}$$

Here

$$\bar{h}_{ij}(\vec{y}) = 4\delta_{ij}(1 - \vec{y}^2)^{-2}, \tag{3.10}$$

$i, j = 1, \ldots, N-1$, are components of the canonical metric on the $(N-1)$-dimensional Lobachevsky space $H^{N-1} = D^{N-1} \equiv \{\vec{y} : |\vec{y}| < 1\}$.

The "wall" potential $V(\vec{y}, B)$ in (3.9)

$$\begin{aligned} V(\vec{y}, B) \equiv \quad & 0, \quad \vec{y} \in B, \\ & +\infty, \quad \vec{y} \in D^{N-1} \setminus B, \end{aligned} \tag{3.11}$$

corresponds to the open domain (billiard)

$$B = \bigcap_{s \in S_+} B_s \subset D^{N-1}, \tag{3.12}$$

where

$$B_s = \{\vec{y} \in D^{N-1} || \vec{y} - \vec{v}^s| > r_s\}, \tag{3.13}$$

and

$$\vec{v}_s = -\vec{u}^s / u_0^s, \qquad r_s = \sqrt{(\vec{v}_s)^2 - 1}, \tag{3.14}$$

$(|\vec{v}_s| > 1)$ $s \in S_+$.

The boundary of the billiard is formed by certain parts of $m_+ = |S_+|$ $(N-2)$-dimensional spheres with centers in points $\vec{v}_s$ and radii r_s, $s \in S_+$.

When $S_+ \neq \emptyset$ the Lagrangian (3.9) describes a motion of a particle of unit mass, moving in the $(N-1)$-dimensional billiard $B \subset D^{N-1}$ (see (3.12)). The geodesic motion in B corresponds to a "Kasner epoch" while the reflection from the boundary corresponds to the change of Kasner epochs.

The billiard B has an infinite volume: vol$B = +\infty$ if and only if there are open zones at the infinite sphere $|\vec{y}| = 1$. After a finite number of reflections from the boundary a particle moves towards one of these open zones. In this case for a corresponding cosmological model we get the "Kasner-like" behavior in the limit $t \to -\infty$ [34].

When vol$B < +\infty$ we get a never ending oscillating behaviour near the singularity.

In [31] the following simple geometric criterion for the finiteness of volume of B was proposed.

Proposition 1 [31]. *The billiard B (3.12) has a finite volume if and only if point-like sources of light located at the points $\vec{v}_s$ $s \in S_+$ (see (3.14)) illuminate the unit sphere S^{N-2}.*

There exists a topological bound on a number of point-like sources m_+ illuminating the sphere S^{N-2} [41]:

$$m_+ \geq N. \tag{3.15}$$

Due to this restriction the number of exponential terms in potential obeying (3.2) $m_+ = |S_+|$ should at least exceed the value $N = n + l$ for the existence of oscillating behaviour near the singularity.

Description in terms of Kasner-like parameters.

For zero potential $V_\varphi = 0$ we get a Kasner-like solutions

$$g = w d\tau \otimes d\tau + \sum_{i=1}^{n} A_i \tau^{2\alpha^i} g^i, \tag{3.16}$$

$$\varphi^\beta = \alpha^\beta \ln \tau + \varphi_0^\beta, \tag{3.17}$$

$$\sum_{i=1}^{n} d_i \alpha^i = \sum_{i=1}^{n} d_i (\alpha^i)^2 + \alpha^\beta \alpha^\gamma h_{\beta\gamma} = 1, \tag{3.18}$$

where $A_i > 0$ and φ_0^β are constants, $i = 1, \ldots, n$; $\beta, \gamma = 1, \ldots, l$.

Let $\alpha = (\alpha^A) = (\alpha^i, \alpha^\gamma)$ obey the relations

$$U^s(\alpha) = U^s_A \alpha^A = \sum_{i=1}^{n} d_i \alpha^i + \lambda_{a_s \gamma} \alpha^\gamma > 0, \tag{3.19}$$

for all $s \in S_+$, then the field configuration (3.16)-(3.18) is the asymptotical (attractor) solution for a family of (exact) solutions, when $\tau \to +0$.

Relations (3.19) may be easily understood using the following relations

$$\Lambda_s \exp[2\lambda_s(\varphi) + 2\gamma_0(\phi)] = \Lambda_s \exp[2U^s(x)] = C_s \tau^{2U^s(\alpha)} \to 0, \tag{3.20}$$

for $\tau \to +0$, where $C_s \neq 0$ are constants, $s \in S_+$. Other terms in the potential (2.6) are also vanishing near the singularity [29, 30, 31, 34]. Thus, the potential (2.6) asymptotically tends to zero as $\tau \to +0$ and we are led to asymptotical solutions (3.16)-(3.18).

Another way to get the conditions (3.19) is based on the isomorphism between S^{N-2} and the Kasner set (3.18)

$$\alpha^A = e^A_a n^a / q, \quad (n^a) = (1, \vec{n}), \quad \vec{n} \in S^{N-2}. \tag{3.21}$$

Here we use the diagonalizing matrix (e^A_a) and the parameter q defined in the previous section (see (2.29)) [31, 34]. Thus, we come to the following proposition.

Proposition 2. *Billiard B (3.12) has a finite volume if and only if there are no α satisfying the relations (3.18) and (3.19).*

So, we obtained a billiard representation for the model under consideration when the restrictions (3.1) are imposed.

Here we present also useful relations describing the billiard in terms of scalar products

$$\vec{v}_s \vec{v}_{s'} = \frac{\vec{u}^s \vec{u}^{s'}}{u^s_0 u^{s'}_0} = b^{-1} \lambda_{s\alpha} \lambda_{s'\beta} h^{\alpha\beta}, \tag{3.22}$$

$s, s' \in S_+$. They follow from the formulas $\vec{u}^s \vec{u}^{s'} - u^s_0 u^{s'}_0 = (U^s, U^{s'})$ and (2.19).

Proposition 3. *For $n > 1$ billiard B (3.12) has an infinite volume.*

Proof. Due to Proposition 2 it is sufficient to present at least one set of Kasner parameters $\alpha = (\alpha^i, \alpha^\gamma)$ obeying the relations (3.18) and (3.19). As an example of such set one may choose any Kasner set α (obeying (3.18)) with $\alpha^\gamma = 0$, for example, with the following components

$$\alpha^1 = \frac{d_1 \pm \sqrt{R}}{d_1(d_1 + d_2)}, \quad \alpha^2 = \frac{d_2 \mp \sqrt{R}}{d_2(d_1 + d_2)}, \quad \alpha^i = 0 \quad (i > 2). \tag{3.23}$$

where $R = d_1 d_2 (d_1 + d_2 - 1)$. In this case inequalities (3.19) are satisfied, since $U^s(\alpha) = 1$ for all s. The proposition is proved.

Thus, according to Proposition 3, for $n > 1$ we obviously have a "Kasner-like" behavior near the singularity (as $\tau \to 0$). The oscillating behaviour near the singularity is impossible in this case.

Remark 1 (general "collision law"). The set of Kasner parameters $({\alpha'}^A)$ after the collision with the s-th wall ($s \in S_+$) is defined by the Kasner set before the collision (α^A) according to the following formula [39]

$${\alpha'}^A = \frac{\alpha^A - 2U^s(\alpha) U^{sA} (U^s, U^s)^{-1}}{1 - 2U^s(\alpha)(U^s, U^\Lambda)(U^s, U^s)^{-1}}, \tag{3.24}$$

where $U^{sA} = \bar{G}^{AB} U^s_B$, $U^s(\alpha) = U^s_A \alpha^A$ and co-vector U^Λ is defined in (2.30). In the special case of one scalar field and 1-dimensional factor-spaces (i.e. $l = d_i = 1$) this formula was suggested earlier in [35].

4 One factor-space

In this section we consider examples of l-dimensional billiards with finite volumes that occur in the model with l-scalar fields ($l \geq 2$) and one scale factor ($n = 1$). Here we put $h_{\alpha\beta} = \delta_{\alpha\beta}$ and $\vec{\lambda}_s = (\lambda_{s1}, \ldots, \lambda_{sl})$.

Thus, here we deal with the Lagrangian

$$\mathcal{L} = R[g] - \partial_M \vec{\varphi} \partial_N \vec{\varphi} - 2 \sum_{s \in S} \Lambda_s \exp[2\vec{\lambda}\vec{\varphi}]. \tag{4.1}$$

where $\vec{\varphi} = (\varphi^1, \ldots, \varphi^l)$.

In this (one-factor case) the following proposition takes place.

Proposition 4. *For $n = 1$ and $h_{\alpha\beta} = \delta_{\alpha\beta}$ the billiard B (3.12) has a finite volume if and only if point-like sources of light located at the points $b^{-1/2}\vec{\lambda}_s \in \mathbb{R}^l$, $s \in S_+$, ($b = (D-1)/(D-2)$) illuminate the unit sphere S^{l-1}* .

Proof. According to relations (3.22) the set of vectors $b^{-1/2}\vec{\lambda}_s \in \mathbb{R}^l$, $s \in S_+$, is coinciding with the set $\vec{v}_s \in \mathbb{R}^l$, $s \in S_+$, up to $O(l)$-transformation, i.e. there exists orthogonal matrix A: $A^T A = \mathbf{1}$, such that $b^{-1/2}\vec{\lambda}_s = A\vec{v}_s \in \mathbb{R}^l$, $s \in S_+$. Then the Proposition 4 follows from Proposition 1, since the sphere S^{l-1} is illuminated by sources $\vec{v}_s$, $s \in S_+$, if and only if, it is illuminated by sources $b^{-1/2}\vec{\lambda}_s$, $s \in S_+$.

According to relations (3.16)-(3.18) we get the following asymptotical behavior for $\tau \to 0$

$$g_{as} = w d\tau \otimes d\tau + A_1 \tau^{2/(D-1)} g^1, \tag{4.2}$$

$$\vec{\varphi}_{as} = \vec{\alpha}_\varphi \ln \tau + \vec{\varphi}_0, \tag{4.3}$$

$$(\vec{\alpha}_\varphi)^2 = b^{-1} = (D-2)/(D-1). \tag{4.4}$$

Here $\vec{\varphi}_0$ and $\vec{\alpha}_\varphi$ change their values after the reflections from the billiard walls. Thus, here we obtained the oscillating behaviour of scalar fields near the singularity.

Remark 2 ("collision law"). From (3.24) we get the "collision law" relation in this case

$$\vec{\alpha}_\varphi' = \frac{\vec{\alpha}_\varphi - 2(1 + \vec{\lambda}_s \vec{\alpha}_\varphi)(\lambda_s^2 - b)^{-1} \vec{\lambda}_s}{1 + 2(1 + \vec{\lambda}_s \vec{\alpha}_\varphi)(\lambda_s^2 - b)^{-1} b}. \tag{4.5}$$

The Kasner parameter for the scale factor is not changed after the "collision".

$l = 2$ **case.** In the special case of two-component scalar field ($l = 2$), and $m_+ = |S_+| = 3$ (i.e. when three "walls" appear) we find the necessary and sufficient condition for the finiteness of the billiard volume in terms of scalar products of the coupling vectors $\vec{\lambda}_s \in \mathbb{R}^2$, $s \in S_+$.

Proposition 5 . *For $n = 1$, $l = 2$, $h_{\alpha\beta} = \delta_{\alpha\beta}$ and $m_+ = |S_+| = 3$ the billiard B (3.12) has a finite volume if and only if the vectors $\vec{\lambda}_s \in \mathbb{R}^2$, $s \in S_+$, obey the following relations:*

$$b^{-1}\vec{\lambda}_s\vec{\lambda}_{s'} \geq 1 - \sqrt{b^{-1}\vec{\lambda}_s^2 - 1}\sqrt{b^{-1}\vec{\lambda}_{s'}^2 - 1}, \qquad s < s', \tag{4.6}$$

$$\sum_{s<s'} \arccos \frac{\vec{\lambda}_s\vec{\lambda}_{s'}}{|\vec{\lambda}_s||\vec{\lambda}_{s'}|} = 2\pi. \tag{4.7}$$

Proof. According to Proposition 3 we should find the necessary and sufficient conditions for three points located in $\vec{v}_s = b^{-1/2}\vec{\lambda}_s$, $s = 1, 2, 3$, to illuminate the unit circle S^1. Here we put $S_+ = \{1, 2, 3\}$ for simplicity. It may be obtained from a simple geometrical consideration that such conditions may be chosen as the following ones:

$$\theta_{ss'} \leq \arccos \frac{1}{|\vec{v}_s|} + \arccos \frac{1}{|\vec{v}_{s'}|}, \qquad s < s', \tag{4.8}$$

and

$$\theta_{12} + \theta_{23} + \theta_{13} = 2\pi \tag{4.9}$$

where

$$\theta_{ss'} = \arccos \frac{\vec{v}_s\vec{v}_{s'}}{|\vec{v}_s||\vec{v}_{s'}|} \tag{4.10}$$

is the angle between vectors $\vec{v_s}$ and $\vec{v_{s'}}$ $s, s' = 1, 2, 3$. Relation (4.8) means that the angle between two vectors $\vec{v_s}$ and $\vec{v_{s'}}$ should not exceed one half of the sum of two arcs on S^1 illuminated by source of light located in points $\vec{v_s}$ and $\vec{v_{s'}}$ (see Fig. 1).

Relations (4.6) may be obtained from (4.8) by acting on both sides of the inequality by function *cos*. Relation (4.7) is just equivalent to (4.8). Relation (4.9) exclude the situation when points $\vec{v_s}$, $s = 1, 2, 3$, belong to a half-plane with a border-line containing the center of the unit circle. The proposition is proved.

An example of (sub-)compact triangle billiard with a finite area in the Lobachevsky space H^2 is depicted on Fig. 1.

Figure 1. Triangle sub-compact billiard with finite volume for $n = 1$, $l = 2$ and $m_+ = 3$.

In the symmetric case when all $\lambda_s^2 = 4b$ and $\vec{\lambda}_s \vec{\lambda}_{s'} = -2b$ for $s \neq s'$ we get an example of non-(sub)-compact billiard with finite area. Such billiard appears in the well-known Bianchi-IX model, see Fig. 2.

For "quasi-Cartan" matrix defined as $A_{ss'} = 2(U^s, U^{s'})/(U^{s'}, U^{s'})$ we get

$$(A_{ss'}) = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}. \qquad (4.11)$$

This matrix coincides with the Cartan matrix of the hyperbolic of Kac-Moody algebra that is number 7 in classification of [42]. This Kac-Moody algebra is a subalgebra of $A_1^{\wedge\wedge}$ [43] (see also [37, 38]).

5 Discussions

In this paper we have considered the behavior near the singularity of the multidimensional cosmological-type model with n Einstein factor-spaces in the theory with scalar fields and MEP (multiple exponential potential). Using the results from [29, 30, 31, 34] we have obtained the billiard representation on

Figure 2. Triangle billiard coinciding with that of Bianchi-IX model.

multidimensional Lobachevsky space for this cosmological-type model near the singularity.

Here we have shown that for $n > 1$ the oscillating behavior near the singularity is absent, i.e. solutions have an asymptotical Kasner-like behavior.

For one-factor case we have described (in terms of illumination problem) the billiards with finite volume and hence with the oscillating behavior of scalar fields near the singularity.

In the model with two scalar fields and three potential walls we have found the necessary and sufficient conditions (in terms of dilatonic coupling vectors) for triangle billiards to be of finite volume.

Appendix

A Equations of motion

Here we outline for the sake of completeness the equations of motions corresponding to the action (1.1)

$$R_{MN} - \frac{1}{2} g_{MN} R = T_{MN}, \tag{A.1}$$

$$\triangle[g]\varphi^\alpha - \sum_{s\in S} 2\lambda_s^\alpha e^{2\lambda_s(\varphi)}\Lambda_s = 0. \qquad \text{(A.2)}$$

In (A.1)

$$T_{MN} = h_{\alpha\beta}\left(\partial_M\varphi^\alpha\partial_N\varphi^\beta - \frac{1}{2}g_{MN}\partial_P\varphi^\alpha\partial^P\varphi^\beta\right) - V_\varphi g_{MN}. \qquad \text{(A.3)}$$

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